

Intrinsic branching structure within random walk on \mathbb{Z}^*

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Abstract

In this paper, we reveal the branching structure for a non-homogeneous random walk with bounded jumps. The ladder time T_1 , the first hitting time of $[1, \infty)$ by the walk starting from 0, could be expressed in terms of a non-homogeneous multitype branching process. As an application of the branching structure, we prove a law of large numbers of random walk in random environment with bounded jumps and specify the explicit invariant density for the Markov chain of “the environment viewed from the particle”. The invariant density and the limit velocity could be expressed explicitly in terms of the environment.

Keywords: random walk, branching process, random environment, invariant density.

Mathematics Subject Classification: Primary 60J80; secondary 60G50.

1 Introduction

1.1 Background

We study random walk $\{X_n\}$ with bounded jumps in this paper. More precisely, let $\Lambda = \{-L, \dots, R\} \setminus \{0\}$. At position x the walk jumps with probability $\omega_x(l)$ to $x + l$ for $l \in \Lambda$. Of course, $\sum_{l \in \Lambda} \omega_x(l) = 1$ and $\omega_x(l) \geq 0$, $l \in \Lambda$. We call $\{X_n\}$ the (L-R) random walk hereafter.

Let T_1 be the time the walk hits the positive half line $(0, \infty)$ for the first time. As we know, it is a fundamental task to characterize T_1 , which plays an important roles in studying the (L-R) random walk, for example, the recurrence vs transience, law of large numbers, central limits theorem and large deviations principle et al.

Indeed, for T_1 , even some simple questions are difficult to answer. For example, what is the distribution of T_1 ? Or further, what are the moments of T_1 .

For some simple setting, these questions are known but are open for some more general setting.

For simple random walk, that is, $L = R = 1$ and $\omega_x(1) = 1 - \omega_x(-1) = p \in [0, 1]$ for all $x \in \mathbb{Z}$. The distribution of T_1 could be find by Reflection Principle. We refer the reader to Strook [6](2005) for the specific calculation.

For (1-1) random walk with non-homogeneous transition probabilities, that is, ω_x depends on x , it has been revealed in Kesten-Kozlov-Spitzer [5](1975) that T_1 could be expressed in term of a non-homogeneous Galton-Watson branching process. This fact enables them to prove a nice stable limit theorem for nearest neighbor Random Walk in Random Environment (RWRE hereafter). If $\max\{L, R\} > 1$, that is, for the non-nearest neighbor random walk, things get very different.

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For (L-1) random walk, Hong-Wang [3](2010) revealed its branching structure, that is, T_1 could be expressed by a multitype (L -type) branching process. Using the branching structure, Hong-Wang also proved a stable limit theorem, partially generalizing the results, for (1-1) RWRE, of Kesten-Kozlov-Spitzer [5](1975) to (L-1) RWRE which is supposed to be transient to the right.

One may ask naturally what the branching structure for (1-R) random walk is. This question is answered in Hong-Zhang [4](2010). The authors decomposed the random walk path to reveal that T_1 for (1-R) random walk could be expressed by a non-homogeneous $(1 + 2 + \dots + R)$ -type branching process and by this fact they also proved a law of large numbers for (1-R) RWRE by a method known as “the environment viewed from particles”.

One should note that, both the above mentioned [3] and [4] treat the case $\min\{L, R\} = 1$. The walk is requested to be nearest neighbor at least in one side. The main purpose of this paper is to consider the general situation: $\min\{L, R\} \geq 1$. We restrict ourselves to $L = R = 2$ to explain the idea. We reveal the intrinsic branching structure within the (L-R) random walk and give some applications.

Next we define the precise model of (L-R) random walk.

1.2 The model

Fix $L, R \geq 1$. Set $\Lambda = \{-L, \dots, R\} \setminus \{0\}$. For $i \in \mathbb{Z}$, let $\omega_i = (\omega_i(l))_{l \in \Lambda}$ be a probability measure on $i + \Lambda$, that is $\sum_{l \in \Lambda} \omega_i(l) = 1$, and $\omega_i(l) \geq 0$ for all $l \in \Lambda$. Set $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$, which will serve as the transition probabilities of the random walk. Let $\{X_n\}_{n \geq 0}$ be a Markov chain with initial value $X_0 = x$ and transition probabilities

$$P_\omega(X_{n+1} = i + j | X_n = i) = \omega_i(j), \quad j \in \Lambda.$$

We call $\{X_n\}$ the *(L-R) random walk with non-homogeneous transition probabilities*. Throughout this paper, we use P_ω^x to denote the law induced by the random walk $\{X_n\}$.

In the remainder of the paper, in order to avoid the heavy notations, we consider the case $L = 2, R = 2$, that is, the (2-2) random walk. The idea for treating the general (L-R) random walk is basically the same as (2-2) setting. Also except otherwise stated, we always assume the random walk starts from 0.

For the above defined (2-2) random walk $\{X_n\}$, set $T_0 = 0$ and define recursively

$$T_k = \inf[n \geq 0 : X_n > X_{T_{k-1}}]$$

for $k \geq 1$. We call the stopping times $T_k, k \geq 1$, the *ladder times* of the random walk.

Especially one sees by the definition that

$$T_1 = \inf[n \geq 0 : X_n > 0]$$

is the hitting time of $[1, \infty)$ by the walk, and X_{T_1} is also a random variable with two possible values $X_{T_1} = 1$ or $X_{T_1} = 2$. This is the reason why we call $T_k, k \geq 1$, the ladder times. The distribution of X_{T_1} , the exit probabilities of the walk from $(-\infty, 0]$, is also a question we concern below.

The main purpose of this paper is to count exactly how many steps the walk spends to exit successfully from $(-\infty, 0]$ (the total steps of the walk before T_1 .) and give some applications. Next we state the main results.

1.3 The main results

In order to count exactly the steps of the walk before T_1 , we define three types of excursions.

Definition 1.1

a) We call excursions of the form $\{X_k = i, X_{k+1} = i-1, X_{k+2} \leq i-1, \dots, X_{k+l} \leq i-1, X_{k+l+1} \geq i\}$ type- \mathcal{A} excursions at i . Corresponding to the three kinds of possible last step of type- \mathcal{A} excursions at i , say, $\{i-1 \rightarrow i\}$, $\{i-2 \rightarrow i\}$ and $\{i-1 \rightarrow i+1\}$, we classify type- \mathcal{A} excursions at i into three sub-types $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$ and $\mathcal{A}_{i,3}$.

b) We call excursions of the form $\{X_k = i, X_{k+1} = i-2, X_{k+2} \leq i-1, \dots, X_{k+l} \leq i-1, X_{k+l+1} \geq i\}$ type- \mathcal{B} excursions at i . Corresponding to the three kinds of possible last step of type- \mathcal{B} excursions at i , say, $\{i-1 \rightarrow i\}$, $\{i-2 \rightarrow i\}$ and $\{i-1 \rightarrow i+1\}$, we classify type- \mathcal{B} excursions at i into three sub-types $\mathcal{B}_{i,1}$, $\mathcal{B}_{i,2}$ and $\mathcal{B}_{i,3}$.

c) We call excursions of the form $\{X_k = i+1, X_{k+1} = i-1, X_{k+2} \leq i-1, \dots, X_{k+l} \leq i-1, X_{k+l+1} \geq i\}$ type- \mathcal{C} excursions at i . Corresponding to the three kinds of possible last step of type- \mathcal{C} excursions at i , say, $\{i-1 \rightarrow i\}$, $\{i-2 \rightarrow i\}$ and $\{i-1 \rightarrow i+1\}$, we classify type- \mathcal{C} excursions at i into three sub-types $\mathcal{C}_{i,1}$, $\mathcal{C}_{i,2}$ and $\mathcal{C}_{i,3}$.

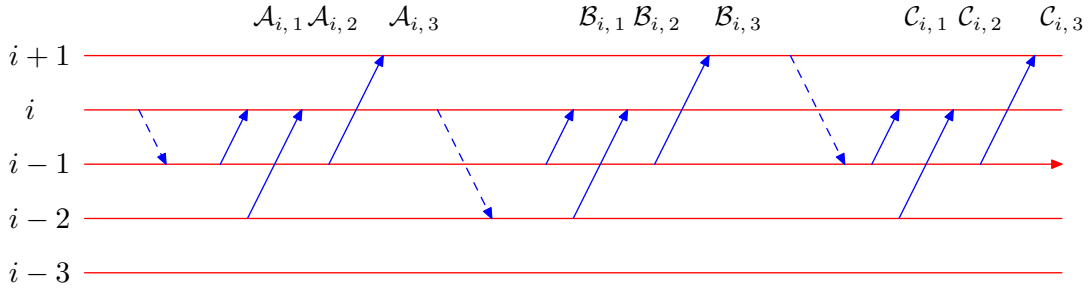


Figure 1. The figure illustrates type \mathcal{A} , \mathcal{B} and \mathcal{C} excursions at i . We draw only the first step and the last step, omitting all things between these two steps. Between these two steps, the walk walks below $i-1$.

Define

$$A_{i,j} = \#\{\mathcal{A}_{i,j} \text{ excursions before } T_1\},$$

$$B_{i,j} = \#\{\mathcal{B}_{i,j} \text{ excursions before } T_1\},$$

$$C_{i,j} = \#\{\mathcal{C}_{i,j} \text{ excursions before } T_1\},$$

for $i \leq 0$ and $j = 1, 2, 3$, where “ $\#\{\}$ ” means the number of the elements in some set.

We aim at counting exactly all steps by the walk before T_1 . For this purpose, define

$$U_i = (A_{i,1}, A_{i,2}, A_{i,3}, B_{i,1}, B_{i,2}, B_{i,3}, C_{i,1}, C_{i,2}, C_{i,3}), \quad (1)$$

being the the total numbers of different excursions at i before time T_1 . Then we have the following fact.

Theorem 1.1 Suppose that $\limsup_{n \rightarrow \infty} X_n = \infty$. Then

$$T_1 = 1 + \sum_{i \leq 0} U_i(2, 2, 1, 1, 1, 0, 2, 2, 1)^T, \quad (2)$$

where for vector $\mathbf{v} \in \mathbb{R}^9$, \mathbf{v}^T denotes the transposition of \mathbf{v} .

The proof of Theorem 1.1 will be given in Section 4 below.

Remark 1.1 *Because the (2-2) random walk we considering is non-nearest neighbor, one can not give the exact distribution of T_1 in general. But we find in this paper that the process $\{U_i\}_{i \leq 0}$ defined in (1) is a non-homogeneous multitype branching process. This fact together with (2) enables us to study T_1 by the properties of branching processes.*

We attach first an ancestor to the branching process. The walk starts from 0. But before T_1 , there is no jump down from above 1 to 0 by the walk. One can imagine that there is a step by the walk from 1 to 0 before it starts from 0 (One can also imagine that this step is from 2 to 0. But this makes no difference.), that is, set $X_{-1} = 1$. Adding this imaginary step, the path $\{X_{-1} = 1, X_0 = 0, X_1, \dots, X_{T_1}\}$ forms a type- \mathcal{A} excursion at 1 such that with probability 1,

$$A_{1,1} + A_{1,2} + A_{1,3} = 1.$$

Then one defines U_1 as in (1). But since there is no $\mathcal{B}_{1,j}$ and $\mathcal{C}_{1,j}$, $j = 1, 2, 3$, excursions, U_1 has only three possible values, that is, $U_1 = \mathbf{e}_1$, $U_1 = \mathbf{e}_2$ or $U_1 = \mathbf{e}_3$.

We can treat U_1 as some particle immigrates in the system and call it “immigration” throughout.

Theorem 1.2 *Suppose that $\limsup_{n \rightarrow \infty} X_n = \infty$. Then $\{U_i\}_{i \leq 1}$ is a 9-type non-homogeneous branching processes with immigration distribution as in (14), (15) and (16) below, and offsprings distributions as in (17-20), (22-25), (26-29) and (30-34) below.*

The proof of Theorem 1.2 will be given in the long Section 3 below.

Remark 1.2 1) While $\omega_i(-2) = 0$ for all $i \in \mathbb{Z}$, Theorem 1.2 reveals the branching structure of (1-2) random walk and while $\omega_i(2) = 0$ for all $i \in \mathbb{Z}$, Theorem 1.2 reveals the branching structure of (2-1) random walk. Therefore the branching structure in Theorem 1.2 contains both the branching structures of Hong-Wang [3] and Hong-Zhang [4]. For the details, see Remark 4.1 below.

2) The authors found that, one could simplify the branching structure. In fact, a 6-type branching process is enough to count exactly all steps by the walk before T_1 . However we still use a 9-type branching process, since it is more understandable and each of the 9 types of particles corresponds to specific jump of the walk. For details, see also Remark 4.1 below.

We give an example to test the branching structure (the above Theorem 1.1 and Theorem 1.2) in Section 5. We consider the degenerated ω , that is, for all $i \in \mathbb{Z}$, $\omega_i(1) = p_1$, $\omega_i(2) = p_2$, $\omega_i(-1) = q_1$ and $\omega_i(-2) = q_2$ with $p_1 + p_2 + q_1 + q_2 = 1$, $p_1, p_2, q_1 \geq 0$ and $q_2 > 0$. In this case, let

$$M = \begin{pmatrix} -\frac{q_1+q_2}{q_2} & \frac{p_1+p_2}{q_2} & \frac{p_2}{q_2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Suppose that $E_\omega^0(X_1) = p_1 + 2p_2 - q_1 - 2q_2 > 0$ implying that $\lim_{n \rightarrow \infty} X_n = \infty$ and let f, g, h be the three eigenvalues of M such that $|f| > |g| > |h|$. Then we show that

$$P_\omega^0(X_{T_1} = 1) = 1 + h \text{ and } P_\omega^0(X_{T_1} = 2) = -h,$$

where $h < 0$ follows from $p_1 + 2p_2 - q_1 - 2q_2 > 0$ (See Section 5 for details.). Then one has that

$$E_\omega^0(X_{T_1}) = 1 - h. \quad (3)$$

On the other hand, as ω is degenerated, one follows from Theorem 1.1 and Theorem 1.2 that

$$E_\omega^0(T_1) = 1 + \sum_{i \leq 0} u_1 Q^{i+1}(2, 2, 1, 1, 1, 0, 2, 2, 1)^T,$$

where Q is the mean offspring matrix of a homogeneous multitype branching process (See (48) below.) and u_1 is the mean of the immigration (See (37) below.). But by the Ward Equation, one has that

$$E_\omega^0(X_{T_1}) = E_\omega^0(T_1)E_\omega^0(X_1) = \left(1 + \sum_{i \leq 0} u_1 Q^{i+1}(2, 2, 1, 1, 1, 0, 2, 2, 1)^T\right)(p_1 + 2p_2 - q_1 - 2q_2). \quad (4)$$

We test (with the help of Matlab) that the right-most-hand sides of (3) and (4) are the same.

At last, as application of the branching structure, we prove a law of large numbers for transient (2-2) random walk in ergodic random environment by a method known as “the environment viewed from particles”.

We first define (2-2) random walk in random environment. Let Ω be the collection of $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ and \mathcal{F} be the Borel σ -algebra on Ω . Define the shift operator on Ω by

$$(\theta\omega)_i = \omega_{i+1}.$$

Let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) making $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ an ergodic system. The so-called random environment is a random element $\omega \in \Omega$ chosen according to the probability \mathbb{P} .

The (2-2) random walk $\{X_n\}_{n \geq 0}$ in random environment ω is define to be a Markov chain with initial value $X_0 = x$ and transition probabilities

$$P_\omega(X_{n+1} = i + j | X_n = i) = \omega_i(j), \quad j \in \{1, 2, -1, -2\}.$$

The measure P_ω^x induced by $\{X_n\}$ on $(\mathbb{Z}^\mathbb{N}, \mathcal{G})$, with \mathcal{G} the Borel σ -algebra, is called the quenched probability and the probability P^x defined on $(\mathbb{Z}^\mathbb{N}, \mathcal{G})$ by the relation

$$P^x(B) = \int P_\omega^x(B) \mathbb{P}(d\omega), \quad B \in \mathcal{G}$$

is called the annealed probability.

We show the following law of large numbers.

Theorem 1.3 *Suppose that $E^0(T_1) < \infty$. Then one has that*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = V_P,$$

for some V_P . Moreover

$$V_P = \frac{E_P\left(\Pi(\omega)(2\omega_0(-2) + \omega_0(-1) + \omega_0(1) + 2\omega_0(2))\right)}{E_P(D(\omega))},$$

where $\Pi(\omega)$ and $D(\omega)$ are defined in (57) and (59) below.

Remark 1.3 (1) The role the branching structure plays is to give the invariant density $\Pi(\omega)$ explicitly in terms of the environment ω .

(2) In Brémont [2], the author also proved a law of large numbers for (L-R) random walk in random environment, see Theorem 1.10 therein. But Brémont [2] did not give the specific form of the velocity V_P . The branching structure enables us to give the invariant density $\Pi(\omega)$ explicitly, so that we can give the velocity V_P explicitly.

2 Exit probability from $[a+1, b-1]$

In this section we calculate the exit probabilities of the walk from certain interval (a, b) .

Fix $a < b$. Let $\partial^+[a, b] = \{b, b+1\}$ and $\partial^-[a, b] = \{a, a-1\}$ be the positive and negative boundaries of $[a, b]$ correspondingly. For $k \in [a-1, b+1]$, $\zeta \in \partial^+[a, b] \cup \partial^-[a, b]$, define

$$\mathcal{P}_k(a, b, \zeta) = P_\omega^k(\text{the walk exits the interval } [a+1, b-1] \text{ at } \zeta).$$

For simplicity, we write $\mathcal{P}_k(a, b, \zeta)$ as $\mathcal{P}_k(\zeta)$ temporarily. Define

$$M_k = \begin{pmatrix} -\frac{\omega_k(-1)+\omega_k(-2)}{\omega_k(-2)} & \frac{\omega_k(1)+\omega_k(2)}{\omega_k(-2)} & \frac{\omega_k(2)}{\omega_k(-2)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and set $\Pi_r^s = M_r \cdots M_s$.

Proposition 2.1 Suppose that $\omega_i(-2) > 0$ for all $i \in \mathbb{Z}$. One has that

$$\left\{ \begin{aligned} \mathcal{P}_{b-1}(b) &= \frac{\mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_2 - \mathbf{e}_3]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right) - \mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_2 - \mathbf{e}_3]^T}{\mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right)}, \\ \mathcal{P}_{b-2}(b) &= \frac{\mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_2 - \mathbf{e}_3]^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_2 - \mathbf{e}_3]^T}{\mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right)}, \end{aligned} \right. \quad (5)$$

and

$$\left\{ \begin{aligned} \mathcal{P}_{b-1}(b+1) &= \frac{\mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_3^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right) - \mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_3^T}{\mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right)}, \\ \mathcal{P}_{b-2}(b+1) &= \frac{\mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_3^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_3^T}{\mathbf{e}_1 \Pi_{a+1}^{b-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 \Pi_{a+1}^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right)}. \end{aligned} \right. \quad (6)$$

Remark 2.1 1) If $\omega_i(-2) = 0$ for all i , then the random walk degenerates to (1-2) random walk. The exit probabilities could be calculated analogously as in Brémont [1]. 2) If $\omega_i(2) = 0$ for all $i \in \mathbb{Z}$, then the random walk degenerates to (2-1) random walk. For the walk transient to the right, the exit probability $\mathcal{P}_\omega^k(-\infty, a, a) = 1$ for all $k \leq a - 1$.

Proof of Proposition 2.1:

One follows from the Markov property that

$$\mathcal{P}_k(\zeta) = \omega_k(2)\mathcal{P}_{k+2}(\zeta) + \omega_k(1)\mathcal{P}_{k+1}(\zeta) + \omega_k(-1)\mathcal{P}_{k-1}(\zeta) + \omega_k(-2)\mathcal{P}_{k-2}(\zeta),$$

which leads to

$$\begin{aligned} (\mathcal{P}_{k-1} - \mathcal{P}_{k-2})(\zeta) &= \frac{\omega_k(-1) + \omega_k(-2)}{\omega_k(-2)}(\mathcal{P}_k - \mathcal{P}_{k-1})(\zeta) \\ &\quad + \frac{\omega_k(1) + \omega_k(2)}{\omega_k(-2)}(\mathcal{P}_{k+1} - \mathcal{P}_k)(\zeta) + \frac{\omega_k(2)}{\omega_k(-2)}(\mathcal{P}_{k+2} - \mathcal{P}_{k+1})(\zeta). \end{aligned} \quad (7)$$

Writing the equation (7) in the matrix form, one has that

$$\begin{pmatrix} -\frac{\omega_k(-1) + \omega_k(-2)}{\omega_k(-2)} & \frac{\omega_k(1) + \omega_k(2)}{\omega_k(-2)} & \frac{\omega_k(2)}{\omega_k(-2)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} (\mathcal{P}_k - \mathcal{P}_{k-1})(\zeta) \\ (\mathcal{P}_{k+1} - \mathcal{P}_k)(\zeta) \\ (\mathcal{P}_{k+2} - \mathcal{P}_{k+1})(\zeta) \end{pmatrix} = \begin{pmatrix} (\mathcal{P}_{k-1} - \mathcal{P}_{k-2})(\zeta) \\ (\mathcal{P}_k - \mathcal{P}_{k-1})(\zeta) \\ (\mathcal{P}_{k+1} - \mathcal{P}_k)(\zeta) \end{pmatrix}. \quad (8)$$

Define

$$V_k(\zeta) = \begin{pmatrix} (\mathcal{P}_{k-1} - \mathcal{P}_{k-2})(\zeta) \\ (\mathcal{P}_k - \mathcal{P}_{k-1})(\zeta) \\ (\mathcal{P}_{k+1} - \mathcal{P}_k)(\zeta) \end{pmatrix}.$$

Then equation (8) becomes

$$V_k(\zeta) = M_k V_{k+1}(\zeta). \quad (9)$$

We note that the equation $V_k(\zeta) = M_k V_{k+1}$ makes sense for $k \in (a, b]$. Since $\mathcal{P}_{a-2}(a, b, \zeta)$ has no sense, so is $V_a(\zeta)$. Since $\mathcal{P}_b(a, b, b) = 1$, $\mathcal{P}_b(a, b, b+1) = 0$, and $\mathcal{P}_{b+1}(a, b, b) = 0$, $\mathcal{P}_{b+1}(a, b, b+1) = 1$, then

$$V_b(b) = \begin{pmatrix} (\mathcal{P}_{b-1} - \mathcal{P}_{b-2})(b) \\ (\mathcal{P}_b - \mathcal{P}_{b-1})(b) \\ (\mathcal{P}_{b+1} - \mathcal{P}_b)(b) \end{pmatrix} = \begin{pmatrix} (\mathcal{P}_{b-1} - \mathcal{P}_{b-2})(b) \\ 1 - \mathcal{P}_{b-1}(b) \\ -1 \end{pmatrix}$$

and

$$V_b(b+1) = \begin{pmatrix} (\mathcal{P}_{b-1} - \mathcal{P}_{b-2})(b+1) \\ (\mathcal{P}_b - \mathcal{P}_{b-1})(b+1) \\ (\mathcal{P}_{b+1} - \mathcal{P}_b)(b+1) \end{pmatrix} = \begin{pmatrix} (\mathcal{P}_{b-1} - \mathcal{P}_{b-2})(b+1) \\ -\mathcal{P}_{b-1}(b+1) \\ 1 \end{pmatrix}.$$

Substituting to (9) one has that

$$\mathbf{e}_1 V_k(b) = (\mathcal{P}_{k-1} - \mathcal{P}_{k-2})(b) = \mathbf{e}_1 M_k \cdots M_{b-1} (\mathcal{P}_{b-1}(b)[\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b)\mathbf{e}_1 + [\mathbf{e}_2 - \mathbf{e}_3])^T.$$

Summing from k to $b-1$, it follows that

$$(\mathcal{P}_{b-2} - \mathcal{P}_{k-2})(b) = \sum_{l=k}^{b-1} \mathbf{e}_1 M_l \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b)[\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b)\mathbf{e}_1 + [\mathbf{e}_2 - \mathbf{e}_3] \right)^T. \quad (10)$$

Note that $\mathcal{P}_a(a, b, b) = 0$. Then

$$\mathcal{P}_{b-2}(b) = \sum_{l=a+2}^{b-1} \mathbf{e}_1 M_l \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b) [\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b) \mathbf{e}_1 + [\mathbf{e}_2 - \mathbf{e}_3] \right)^T.$$

On the other hand, since

$$0 = (\mathcal{P}_a - \mathcal{P}_{a-1})(b) = \mathbf{e}_1 M_{a+1} \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b) [\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b) \mathbf{e}_1 + [\mathbf{e}_2 - \mathbf{e}_3] \right)^T$$

then

$$\begin{cases} \mathbf{e}_1 M_{a+1} \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b) [\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b) \mathbf{e}_1 + [\mathbf{e}_2 - \mathbf{e}_3] \right)^T = 0, \\ \mathcal{P}_{b-2}(b) = \sum_{l=a+1}^{b-1} \mathbf{e}_1 M_l \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b) [\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b) \mathbf{e}_1 + [\mathbf{e}_2 - \mathbf{e}_3] \right)^T. \end{cases} \quad (11)$$

Solving (11), one gets (5).

Also, for $a+1 \leq k \leq b-3$, one has from (10) that

$$\begin{aligned} \mathcal{P}_k(b) = & \mathcal{P}_{b-2}(b) \left(1 + \sum_{l=k+2}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right) \\ & - \mathcal{P}_{b-1}(b) \sum_{l=k+2}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \sum_{l=k+2}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_2 - \mathbf{e}_3]^T. \end{aligned}$$

Next we calculate $\mathcal{P}_{b-1}(b+1)$ and $\mathcal{P}_{b-2}(b+1)$. Similarly as (11), one has that

$$\begin{cases} \mathbf{e}_1 M_{a+1} \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b+1) [\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b+1) \mathbf{e}_1 + \mathbf{e}_3 \right)^T = 0, \\ \mathcal{P}_{b-2}(b+1) = \sum_{l=a+1}^{b-1} \mathbf{e}_1 M_l \cdots M_{b-1} \left(\mathcal{P}_{b-1}(b+1) [\mathbf{e}_1 - \mathbf{e}_2] - \mathcal{P}_{b-2}(b+1) \mathbf{e}_1 + \mathbf{e}_3 \right)^T \end{cases}$$

which leads to (6).

Also for $a+1 \leq k \leq b-3$, one has that

$$\begin{aligned} \mathcal{P}_k(b+1) = & \mathcal{P}_{b-2}(b+1) \left(1 + \sum_{l=k+2}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_1^T \right) \\ & - \mathcal{P}_{b-1}(b+1) \sum_{l=k+2}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} [\mathbf{e}_1 - \mathbf{e}_2]^T - \sum_{l=k+2}^{b-1} \mathbf{e}_1 \Pi_l^{b-1} \mathbf{e}_3^T. \end{aligned}$$

□

3 Path decomposition—Proofs of Theorem 1.2

Throughout this section, we always assume that $\limsup_{n \rightarrow \infty} X_n = \infty$. That is, the walk $\{X_n\}$ is transient to ∞ or recurrent. The notation $\{i \rightarrow j\}$ will be always used to denote a jump (a step) by the walk from i to j .

Define

$$T_1 = \inf\{n \geq 0 : X_n > 0\},$$

the hitting time of $[1, \infty)$. The purpose of this section is to count exactly all steps by the walk before T_1 .

3.1 The excursions, corresponding probabilities and immigration distributions

From Proposition 2.1 and Remark 2.1, we see that the exit probabilities of the walk from certain interval (a, b) could be expressed in terms of ω . So we always assume that all these exit probabilities are already known in the remainder of the paper.

For $k \leq i < j$, denote

$$f_k(i, j) = P_\omega^k(\text{the walk hits } (i, \infty) \text{ from below at } j).$$

We remark that for (2-2) random walk, in the definition of $f_k(i, j)$, the term j only takes values in $\{i+1, i+2\}$. With the notations of Section 2,

$$f_k(i, i+1) = \mathcal{P}_k(-\infty, i+1, i+1) \text{ and } f_k(i, i+2) = \mathcal{P}_k(-\infty, i+1, i+2).$$

Next we analyze the path of the walk.

Firstly, we consider a special excursion, which will be called type- \mathcal{A} excursion latter, of the walk.

Definition 3.1 We call excursions of the form $\{X_k = i, X_{k+1} = i-1, X_{k+2} \leq i-1, \dots, X_{k+l} \leq i-1, X_{k+l+1} \geq i\}$ type- \mathcal{A} excursions at i . Corresponding to the three kinds of possible last step of type- \mathcal{A} excursions i , say, $\{i-1 \rightarrow i\}$, $\{i-2 \rightarrow i\}$ and $\{i-1 \rightarrow i+1\}$, we classify type- \mathcal{A} excursions at i into three sub-types $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$ and $\mathcal{A}_{i,3}$.

An excursion will be also called a particle some times in the remainder of the paper.

Note that a type- \mathcal{A} particle at i begins when the walk jumps down from i to $i-1$. After that the walk runs in $(-\infty, i-1]$. At last the walk hits $[i, \infty)$ at some j and the excursion goes to end.

Next we define some indexes $\alpha_{i,1}$, $\alpha_{i,3}$ and $\alpha_{i,2}$ correspondingly to $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,3}$, and $\mathcal{A}_{i,2}$. Let

$$\begin{aligned} \alpha_{i,1} &:= \omega_i(-1) \sum_{n,m \geq 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1)f_{i-2}(i-2, i-1)]^n [\omega_{i-1}(-2)f_{i-3}(i-2, i-1)]^m \omega_{i-1}(1), \\ \alpha_{i,3} &:= \omega_i(-1) \sum_{n,m \geq 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1)f_{i-2}(i-2, i-1)]^n [\omega_{i-1}(-2)f_{i-3}(i-2, i-1)]^m \omega_{i-1}(2), \\ \alpha_{i,2} &:= \omega_i(-1) - \alpha_{i,1} - \alpha_{i,3}. \end{aligned}$$

Note that $\alpha_{i,1}$ differs from $\alpha_{i,3}$ only in the last term of the product. Therefore we explain only the meaning of $\alpha_{i,1}$. The first term $\omega_i(-1)$ is transition probability of the first step of the excursion $\mathcal{A}_{i,1}$ from i to $i-1$. The last term $\omega_{i-1}(1)$ is transition probability of the last step of the excursion $\mathcal{A}_{i,1}$ from $i-1$ to i . The summation in the center indicates all events occurring between the first step and the last step. In details, before the last step happens, n steps of the form $\{i-1 \rightarrow i-2\}$ and m steps of the form $\{i-1 \rightarrow i-3\}$ occur and the total number of possible combinations of these $m+n$ steps is $\binom{n+m}{m} = \frac{(n+m)!}{n!m!}$. The term $\omega_{i-1}(-1)f_{i-2}(i-2, i-1)$ means that, with probability $\omega_{i-1}(-1)$, a step

$\{i-1 \rightarrow i-2\}$ occurs, and starting from $i-2$, with probability $f_{i-2}(i-2, i-1)$, it hits $[i-1, \infty)$ at $i-1$. The term $\omega_{i-1}(-2)f_{i-3}(i-2, i-1)$ could be explained analogously.

In fact, $\alpha_{i,2}$ could be defined similarly as $\alpha_{i,1}$ and $\alpha_{i,3}$. But the definition is tedious. We note that after the walk jumps down with probability $\omega_i(-1)$ from i to $i-1$, then starting from $i-1$, it hits $[i, \infty)$ with probability 1. Then the summation of $\alpha_{i,1}$, $\alpha_{i,2}$, and $\alpha_{i,3}$ should be $\omega_i(-1)$. So we define

$$\alpha_{i,2} := \omega_i(-1) - \alpha_{i,1} - \alpha_{i,3}.$$

Some easy calculation shows that

$$\begin{aligned}\alpha_{i,1} &= \frac{\omega_i(-1)\omega_{i-1}(1)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}, \\ \alpha_{i,3} &= \frac{\omega_i(-1)\omega_{i-1}(2)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}, \\ \alpha_{i,2} &= \frac{\omega_i(-1)[1 - \omega_{i-1}(1) - \omega_{i-1}(2) - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)]}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}.\end{aligned}$$

Secondly, we describe another excursion, that is, type- \mathcal{B} excursion.

Definition 3.2 We call excursions of the form $\{X_k = i, X_{k+1} = i-2, X_{k+2} \leq i-1, \dots, X_{k+l} \leq i-1, X_{k+l+1} \geq i\}$ type- \mathcal{B} excursions at i . Corresponding to the three kinds of possible last step of type- \mathcal{B} excursions at i , say, $\{i-1 \rightarrow i\}$, $\{i-2 \rightarrow i\}$ and $\{i-1 \rightarrow i+1\}$, we classify type- \mathcal{B} excursions at i into three sub-types $\mathcal{B}_{i,1}$, $\mathcal{B}_{i,2}$ and $\mathcal{B}_{i,3}$.

Note that a type- \mathcal{B} excursion begins when the walk jumps down from i to $i-2$. After that the walk runs in $(-\infty, i-1]$. At last the walk hits $[i, \infty)$ at some j and the excursion goes to end.

Next we define some indexes $\beta_{i,1}$, $\beta_{i,2}$ and $\beta_{i,3}$ corresponding to $\mathcal{B}_{i,1}$, $\mathcal{B}_{i,3}$, and $\mathcal{B}_{i,2}$. Let

$$\begin{aligned}\beta_{i,1} &:= \omega_i(-2)f_{i-2}(i-2, i-1) \\ &\quad \times \sum_{n,m \geq 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1)f_{i-2}(i-2, i-1)]^n [\omega_{i-1}(-2)f_{i-3}(i-2, i-1)]^m \omega_{i-1}(1), \\ \beta_{i,3} &:= \omega_i(-2)f_{i-2}(i-2, i-1) \\ &\quad \times \sum_{n,m \geq 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1)f_{i-2}(i-2, i-1)]^n [\omega_{i-1}(-2)f_{i-3}(i-2, i-1)]^m \omega_{i-1}(2), \\ \beta_{i,2} &:= \omega_i(-2) - \beta_{i,1} - \beta_{i,3}.\end{aligned}\tag{12}$$

$\beta_{i,1}$ differs from $\alpha_{i,1}$ only in the term $\omega_i(2)f_{i-2}(i-2, i-1)$. We explain only this term. Starting from i , with probability $\omega_i(-2)$, the walk jumps down from i to $i-2$ and the excursion begins. Recall that $\beta_{i,1}$ is the index corresponding to a $\mathcal{B}_{i,1}$ excursion. Since a $\mathcal{B}_{i,1}$ excursion ends with a jump $\{i-1 \rightarrow i\}$, after visiting $i-2$, it will reach $i-1$ from below before it comes to end. The sum of products of transition probabilities of all possible paths from $i-2$ to hit $i-1$ from below is $f_{i-2}(i-2, i-1)$.

One follows from (12) that

$$\begin{aligned}\beta_{i,1} &= \frac{\omega_i(-2)f_{i-2}(i-2, i-1)\omega_{i-1}(1)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}, \\ \beta_{i,3} &= \frac{\omega_i(-2)f_{i-2}(i-2, i-1)\omega_{i-1}(2)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)},\end{aligned}$$

$$\beta_{i,2} = \frac{\omega_i(-2)[1 - (\omega_{i-1}(-1) + \omega_{i-1}(1) + \omega_{i-1}(2))f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)]}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}.$$

At last, we define the third kind of excursion, type- \mathcal{C} excursion.

Definition 3.3 We call excursions of the form $\{X_k = i+1, X_{k+1} = i-1, X_{k+2} \leq i-1, \dots, X_{k+l} \leq i-1, X_{k+l+1} \geq i\}$ type- \mathcal{C} excursions at i . Corresponding to the three kinds of possible last step of type- \mathcal{C} excursions i , say, $\{i-1 \rightarrow i\}$, $\{i-2 \rightarrow i\}$ and $\{i-1 \rightarrow i+1\}$, we classify type- \mathcal{C} excursions at i into three sub-types $\mathcal{C}_{i,1}$, $\mathcal{C}_{i,2}$ and $\mathcal{C}_{i,3}$.

Note that a type- \mathcal{C} excursion at i begins when the walk jumps down from $i+1$ to $i-1$. After that the walk runs in $(-\infty, i-1]$. At last the walk hits $[i, \infty)$ at some j and the excursion goes to end.

Next we define some indexes $\gamma_{i,1}$, $\gamma_{i,2}$ and $\gamma_{i,3}$ corresponding to $\mathcal{C}_{i,1}$, $\mathcal{C}_{i,3}$, and $\mathcal{C}_{i,2}$. Let

$$\begin{aligned} \gamma_{i,1} &:= \omega_{i+1}(-2) \sum_{n,m \geq 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1)f_{i-2}(i-2, i-1)]^n [\omega_{i-1}(-2)f_{i-3}(i-2, i-1)]^m \omega_{i-1}(1), \\ \gamma_{i,3} &:= \omega_{i+1}(-2) \sum_{n,m \geq 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1)f_{i-2}(i-2, i-1)]^n [\omega_{i-1}(-2)f_{i-3}(i-2, i-1)]^m \omega_{i-1}(2), \quad (13) \\ \gamma_{i,2} &:= \omega_{i+1}(-2) - \gamma_{i,1} - \gamma_{i,3}. \end{aligned}$$

A $\mathcal{C}_{i,1}$ excursion differs from an $\mathcal{A}_{i,1}$ excursion only in the first step. The first term in the product of $\gamma_{i,1}$ is $\omega_{i+1}(-2)$. It means that with probability $\omega_{i+1}(-2)$, the walk jumps down from $i+1$ to $i-1$, and the excursion begins. $\gamma_{i,2}$ and $\gamma_{i,3}$ could be understood analogously as $\alpha_{i,2}$ and $\alpha_{i,3}$. We will not repeat them here.

One follows from (13) that

$$\begin{aligned} \gamma_{i,1} &= \frac{\omega_{i+1}(-2)\omega_{i-1}(1)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}, \\ \gamma_{i,3} &= \frac{\omega_{i+1}(-2)\omega_{i-1}(2)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}, \\ \gamma_{i,2} &= \frac{\omega_{i+1}(-2)[1 - \omega_{i-1}(1) - \omega_{i-1}(2) - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)]}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)}. \end{aligned}$$

Define

$$\begin{aligned} A_{i,j} &= \#\{\mathcal{A}_{i,j} \text{ excursions before } T_1\}, \\ B_{i,j} &= \#\{\mathcal{B}_{i,j} \text{ excursions before } T_1\}, \\ C_{i,j} &= \#\{\mathcal{C}_{i,j} \text{ excursions before } T_1\}, \end{aligned}$$

for $i \leq 0$ and $j = 1, 2, 3$.

We aim at counting exactly all steps by the walk before T_1 . For this purpose, define

$$U_i = (A_{i,1}, A_{i,2}, A_{i,3}, B_{i,1}, B_{i,2}, B_{i,3}, C_{i,1}, C_{i,2}, C_{i,3})$$

being the total number of different excursions at i before time T_1 . Next we show by path decomposition that $\{U_i\}_{i \leq 0}$ is a non-homogeneous multitype branching process.



Figure 2. The figure illustrates the immigration of the branching process. Adding the imaginary step $\{1 \rightarrow 0\}$, the path of the walk before T_1 forms an \mathcal{A} excursion at 1. It may be an $\mathcal{A}_{1,1}$, $\mathcal{A}_{1,2}$ or $\mathcal{A}_{1,3}$ excursion

Firstly, the branching process needs an ancestor (some particle immigrating in). The walk starts from 0. But before T_1 , there is no jump down from above 1 to 0 by the walk. One can imagine that there is a step by the walk from 1 to 0 before it starts from 0 (One can also imagine that this step is from 2 to 0. But this makes no difference.), that is, set $X_{-1} = 1$. Adding this imaginary step, the path $\{X_{-1} = 1, X_0 = 0, X_1, \dots, X_{T_1}\}$ forms a type- \mathcal{A} excursion at 1 such that

$$A_{1,1} + A_{1,2} + A_{1,3} = 1.$$

The distributions of $A_{1,1}$, $A_{1,2}$, and $A_{1,3}$ are

$$P_\omega^0(A_{1,1} = 1) = \frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}} = \frac{\omega_0(1)}{1 - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)},$$

$$P_\omega^0(A_{1,3} = 1) = \frac{\alpha_{1,3}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}} = \frac{\omega_0(2)}{1 - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)},$$

and

$$\begin{aligned} P_\omega^0(A_{1,2} = 1) &= \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}} \\ &= \frac{[1 - \omega_0(1) - \omega_0(2) - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)]}{1 - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)}. \end{aligned}$$

The meaning of $P_\omega^0(A_{1,1} = 1)$ is obvious. The total sum of the product of transition probabilities of all excursions of the form $\{X_{-1} = 0, X_0 = 1, X_1, \dots, X_{T_1}\}$ is $\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3} = \omega_1(-1)$ and the total sum of the product of transition probabilities of all possible paths of an $\mathcal{A}_{1,1}$ excursion is $\alpha_{1,1}$. Therefore $P_\omega^0(A_{1,1} = 1) = \frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}$. The values of $P_\omega^0(A_{1,2} = 1)$ and $P_\omega^0(A_{1,3} = 1)$ could be explained analogously.

We can treat the above discussed imaginary excursion as the particle immigrates in the branching system, say, the ancestor (immigration) of the branching process. The immigration laws are

$$P_\omega^0(U_1 = (1, 0, \dots, 0)) = \frac{\omega_0(1)}{1 - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)}, \quad (14)$$

$$P_\omega^0(U_1 = (0, 1, 0, \dots, 0)) = \frac{\omega_0(2)}{1 - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)}, \quad (15)$$

and

$$P_{\omega}^0(U_1 = (0, 0, 1, 0, \dots, 0)) = \frac{[1 - \omega_0(1) - \omega_0(2) - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)]}{1 - \omega_0(-1)f_{-1}(-1, 0) - \omega_0(-2)f_{-2}(-1, 0)}. \quad (16)$$

3.2 Branching mechanisms

After revealing the immigration, we discuss the branching mechanism. Although there are 9 types of particles in the system, many of them share the same offspring distributions.

(a) Offspring distributions of $\mathcal{A}_{i+1,1}$, $\mathcal{A}_{i+1,3}$, $\mathcal{C}_{i+1,1}$ and $\mathcal{C}_{i+1,3}$ particles

For $i \leq 0$, conditioned on $U_{i+1} = (1, 0, \dots, 0)$, that is, $\{A_{i+1,1} = 1, A_{i+1,2} = A_{i+1,3} = B_{i+1,1} = B_{i+1,2} = B_{i+1,3} = C_{i+1,1} = C_{i+1,2} = C_{i+1,3} = 0\}$, we study the distribution of U_i . Note that the first step of excursion $\mathcal{A}_{i+1,1}$ is $\{i+1 \rightarrow i\}$ and the last step is $\{i \rightarrow i+1\}$. All contributions of this excursion to U_i occurs between these two steps. Therefore this particle could give births only to excursions $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$, $\mathcal{B}_{i,1}$, and $\mathcal{B}_{i,2}$. We calculate the probability of the event

$$\{A_{i,1} = a, A_{i,2} = b, B_{i,1} = c, B_{i,2} = d\},$$

that is,

$$\{U_i = (a, b, 0, c, d, 0, 0, 0)\},$$

conditioned on $\{A_{i+1,1} = 1\}$ (Or $U_{i+1} = (1, 0, \dots, 0)$). Indeed, one follows from the strong Markov property that these $a+b+c+d$ excursions at i are independent, and the total number of all possible combinations of those excursions is $\frac{(a+b+c+d)!}{a!b!c!d!}$. Therefore

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 0, 0, 0) | U_{i+1} = (1, 0, \dots, 0)) \\ = \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}). \end{aligned} \quad (17)$$

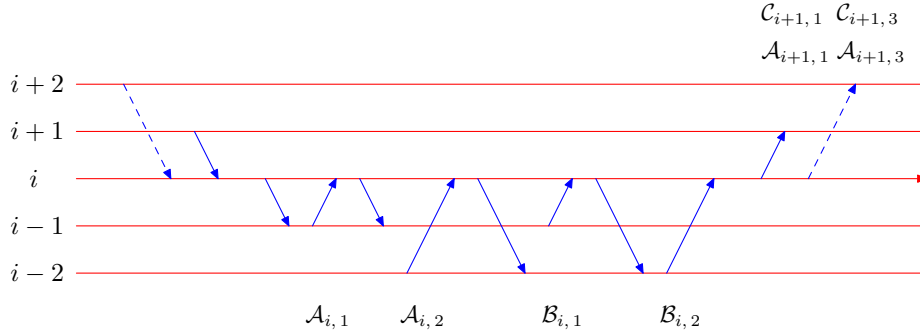


Figure 3. The figure illustrates the offsprings of $\mathcal{A}_{i+1,1}$, $\mathcal{A}_{i+1,3}$, $\mathcal{C}_{i+1,1}$ and $\mathcal{C}_{i+1,3}$ particles. They could only give births to $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$, $\mathcal{B}_{i,1}$ and $\mathcal{B}_{i,2}$ particles.

One notes that $\mathcal{A}_{i+1,3}$, $\mathcal{C}_{i+1,1}$, $\mathcal{C}_{i+1,3}$ and $\mathcal{A}_{i+1,1}$ excursions share a common property, that is, the first step is from above i to i and the last step is from i to above i . Therefore they share the same offspring distribution. So analogously, one has that

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 0, 0, 0) | U_{i+1} = (0, 0, 1, 0, \dots, 0)) \\ = \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}), \end{aligned} \quad (18)$$

$$\begin{aligned}
P_\omega^0(U_i = (a, b, 0, c, d, 0, 0, 0, 0) | U_{i+1} = (0, \dots, 0, 1, 0, 0)) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}),
\end{aligned} \tag{19}$$

$$\begin{aligned}
P_\omega^0(U_i = (a, b, 0, c, d, 0, 0, 0, 0) | U_{i+1} = (0, \dots, 0, 1)) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}).
\end{aligned} \tag{20}$$

(b) Offspring distributions of $\mathcal{A}_{i+1,2}$, and $\mathcal{C}_{i+1,2}$ particles

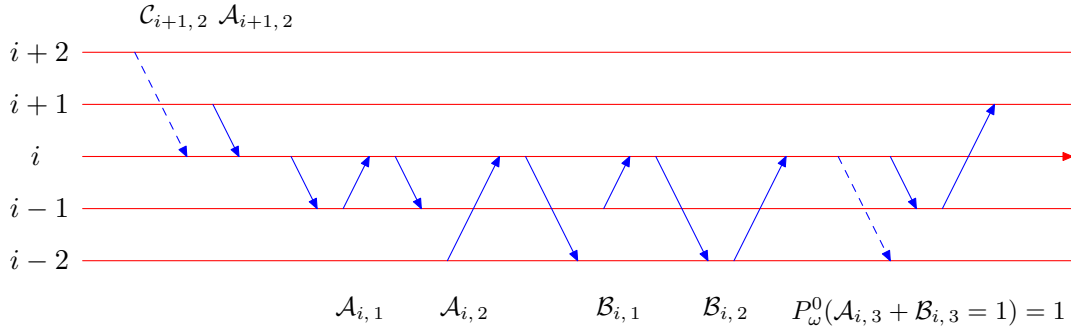


Figure 4. The figure illustrates the offsprings of $\mathcal{A}_{i+1,2}$, $\mathcal{C}_{i+1,2}$. Before the last step $\{i-1 \rightarrow i+1\}$ happens, with probability 1, a $\mathcal{B}_{i,3}$ or $\mathcal{A}_{i,3}$ excursion would be born.

Conditioned on $\{U_{i+1} = (0, 1, 0, \dots, 0)\}$, that is, $\{A_{i+1,2} = 1, A_{i+1,1} = A_{i+1,3} = B_{i+1,1} = B_{i+1,2} = B_{i+1,3} = C_{i+1,1} = C_{i+1,2} = C_{i+1,3} = 0\}$, we discuss the distribution of U_i . Recall that the first step of an $\mathcal{A}_{i+1,2}$ excursion is $\{i+1 \rightarrow i\}$ and the last step is $\{i-1 \rightarrow i+1\}$. Things get delicate because the last step $\{i-1 \rightarrow i+1\}$. Before the last step occurs, the walk must jump down from i , possibly to $i-1$ or $i-2$. If it jumps down from i to $i-1$, it gives birth to an $\mathcal{A}_{i,3}$ particle; if it jumps down from i to $i-2$, it gives birth to a $\mathcal{B}_{i,3}$ particle. That is

$$P_\omega^0(A_{i,3} + B_{i,3} = 1 | U_{i+1} = \mathbf{e}_2) = 1.$$

The sum of the product of the transition probabilities of all possible paths of an excursion $\mathcal{A}_{i,3}$ is $\alpha_{i,3}$, and that of a $\mathcal{B}_{i,3}$ excursion is $\beta_{i,3}$. In this point of view, one concludes that

$$P_\omega^0(A_{i,3} = 1 | U_{i+1} = \mathbf{e}_2) = 1 - P_\omega^0(B_{i,3} = 1 | U_{i+1} = \mathbf{e}_2) = \frac{\alpha_{i,3}}{\alpha_{i,3} + \beta_{i,3}}. \tag{21}$$

Before giving birth to the above discussed particle $\mathcal{B}_{i,3}$ or $\mathcal{A}_{i,3}$, the excursion $\mathcal{A}_{i+1,2}$ may give birth to a number of $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$, $\mathcal{B}_{i,1}$ and $\mathcal{B}_{i,2}$ particles. Once again, one follows by path decomposition and Markov property that, all excursions born to particle $\mathcal{A}_{i+1,2}$ are independent. Therefore one has that

$$\begin{aligned}
P_\omega^0(U_i = (a, b, 1, c, d, 0, 0, 0, 0) | U_{i+1} = (0, 1, 0, \dots, 0)) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\alpha_{i,3}}{\alpha_{i,3} + \beta_{i,3}},
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
P_\omega^0(U_i = (a, b, 0, c, d, 1, 0, 0, 0) | U_{i+1} = (0, 1, 0, \dots, 0)) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\beta_{i,3}}{\alpha_{i,3} + \beta_{i,3}}.
\end{aligned} \tag{23}$$

Also, analogously, the offspring distributions of a $\mathcal{C}_{i+1,2}$ particle are

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 1, c, d, 0, 0, 0, 0) | U_{i+1} = (0, \dots, 0, 1, 0)) \\ = \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\alpha_{i,3}}{\alpha_{i,3} + \beta_{i,3}}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 1, 0, 0, 0) | U_{i+1} = (0, \dots, 0, 1, 0)) \\ = \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\beta_{i,3}}{\alpha_{i,3} + \beta_{i,3}}. \end{aligned} \quad (25)$$

(C) Offspring distributions of $\mathcal{B}_{i+1,1}$, and $\mathcal{B}_{i+1,3}$ particles

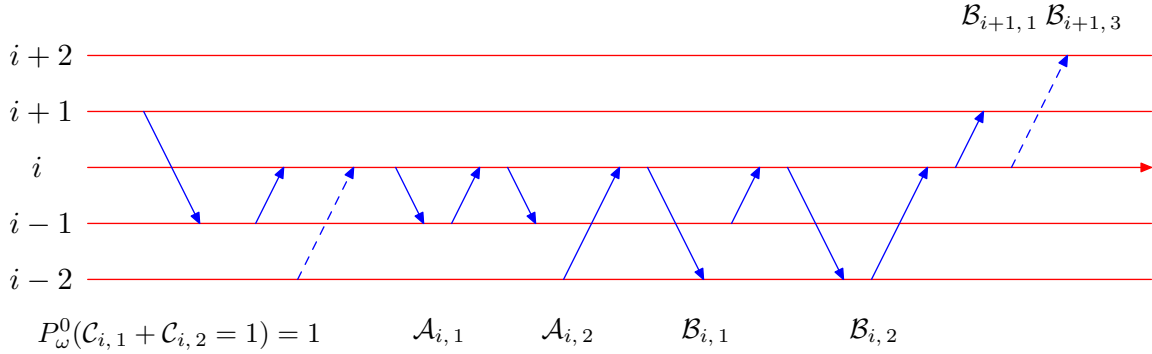


Figure 5. The figure illustrates the offsprings of $\mathcal{B}_{i+1,1}$ and $\mathcal{B}_{i+1,3}$. Since at last, the walk jumps from i to some position above i , before the last step happens, it must return to i from below. Therefore with probability 1, a $\mathcal{C}_{i,1}$ or a $\mathcal{C}_{i,2}$ excursion would be born.

$\mathcal{B}_{i+1,1}$ and $\mathcal{B}_{i+1,3}$ excursions differ from each other only in the last step. But their last steps are both from i to above i . Therefore they have the same offspring distributions. Conditioned on $U_{i+1} = (0, 0, 0, 1, 0, \dots, 0)$, that is, $\{B_{i+1,1} = 1, A_{i+1,1} = A_{i+1,2} = A_{i+1,3} = B_{i+1,2} = B_{i+1,3} = C_{i+1,1} = C_{i+1,2} = C_{i+1,3} = 0\}$, we discuss the offspring distributions of $\mathcal{B}_{i+1,1}$ particle. Note that an excursion $\mathcal{B}_{i+1,1}$ begins with a jump $\{i+1 \rightarrow i-1\}$ and ends with a jump $\{i \rightarrow i+1\}$. So after jumping down from $i+1$ to $i-1$, it must have an excursion returning to i . There are two approaches for the walk to return to i , that is, jumping from $i-1$ to i or jumping from $i-2$ to i . From this point of view, one knows that a $\mathcal{B}_{i+1,1}$ particle gives birth to a $\mathcal{C}_{i,1}$ or $\mathcal{C}_{i,2}$ particle at i with probability 1, and the approach the walk jumping from below i to i determines which particle will be born. Precisely, one has that

$$P_{\omega}^0(C_{i,1} + C_{i,2} = 1 | U_{i+1} = \mathbf{e}_4) = 1.$$

Due to the same reason as (21),

$$P_{\omega}^0(C_{i,1} = 1 | U_{i+1} = \mathbf{e}_4) = 1 - P_{\omega}^0(C_{i,2} = 1 | U_{i+1} = \mathbf{e}_4) = \frac{\gamma_{i,1}}{\gamma_{i,1} + \gamma_{i,2}}.$$

After giving birth to a $\mathcal{C}_{i,1}$ or $\mathcal{C}_{i,2}$ particle, a $\mathcal{B}_{i+1,1}$ particle may give births to certain number of $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$, $\mathcal{B}_{i,1}$ and $\mathcal{B}_{i,2}$ excursions. Markov property implies the independence of those born excursions. One

has that

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 0, 1, 0, 0) | U_{i+1} = (0, 0, 0, 1, 0, \dots, 0)) \\ = \frac{(a + b + c + d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\gamma_{i,1}}{\gamma_{i,1} + \gamma_{i,2}}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 0, 0, 1, 0) | U_{i+1} = (0, 0, 0, 1, 0, \dots, 0)) \\ = \frac{(a + b + c + d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\gamma_{i,2}}{\gamma_{i,1} + \gamma_{i,2}}. \end{aligned} \quad (27)$$

Analogously, the offspring distributions of a $\mathcal{B}_{i+1,3}$ particle are

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 0, 1, 0, 0) | U_{i+1} = (0, \dots, 0, 1, 0, 0, 0)) \\ = \frac{(a + b + c + d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\gamma_{i,1}}{\gamma_{i,1} + \gamma_{i,2}}, \end{aligned} \quad (28)$$

and

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 0, c, d, 0, 0, 1, 0) | U_{i+1} = (0, \dots, 0, 1, 0, 0, 0)) \\ = \frac{(a + b + c + d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{\gamma_{i,2}}{\gamma_{i,1} + \gamma_{i,2}}. \end{aligned} \quad (29)$$

(d) Offspring distribution of $\mathcal{B}_{i+1,2}$ particles

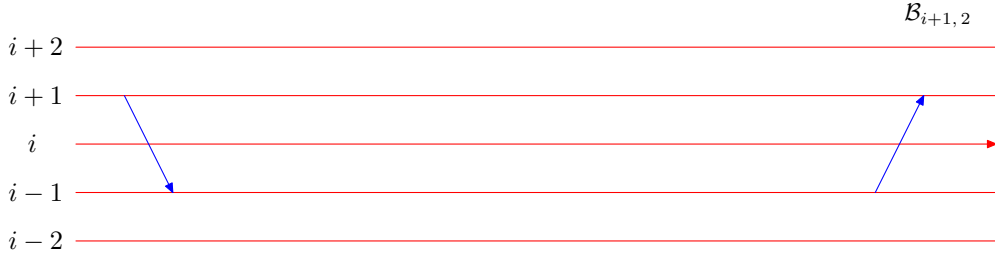


Figure 6a. The figure illustrates the offsprings (case 1) of $\mathcal{B}_{i+1,2}$ excursion. The walk never visited i between the first and the last step. Therefore, only a type $\mathcal{C}_{i,3}$ excursion would be born.

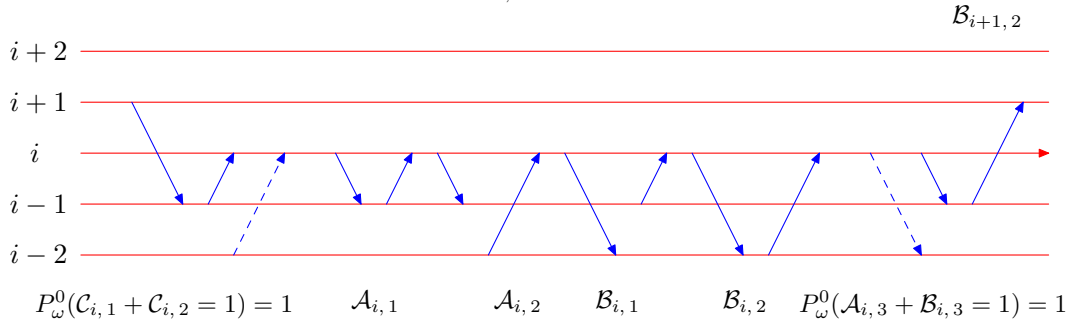


Figure 6b. The figure illustrates the offsprings (case 2) of $\mathcal{B}_{i+1,2}$ excursion. Between the first and the last step, the walk did visit i . Therefore $P_{\omega}^0(\mathcal{C}_{i,1} + \mathcal{C}_{i,2} = 1) = 1$ and $P_{\omega}^0(\mathcal{A}_{i,3} + \mathcal{B}_{i,3} = 1) = 1$.

Next conditioned on $U_{i+1} = (0, 0, 0, 0, 1, 0, 0, 0, 0)$, that is, $\{B_{i+1,2} = 1, A_{i+1,1} = A_{i+1,2} = A_{i+1,3} = B_{i+1,1} = B_{i+1,3} = C_{i+1,1} = C_{i+1,2} = C_{i+1,3} = 0\}$, we consider the offspring distributions of $\mathcal{B}_{i+1,2}$ particles.

Special attention should be paid to $\mathcal{B}_{i+1,2}$ excursions. An excursion $\mathcal{B}_{i+1,2}$ begins with a jump $\{i+1 \rightarrow i-1\}$ and ends with a jump $\{i-1 \rightarrow i+1\}$. The point is whether it visited i between the first and the last step. If it did not visit i before the last step, then the excursion gives birth to a $\mathcal{C}_{i,3}$ particle with probability 1 and generates no any other particle. If it did visit i before the last step, that is, conditioned on $\{C_{i,3} = 0\}$, things get much more complicated. Conditioned on $\{C_{i,3} = 0\}$, after jumping down from $i+1$ to $i-1$, the walk reaches i from below i at least one time, that is, from $i-1$ to i or from $i-2$ to i . Therefore one has that

$$P_{\omega}^0(C_{i,1} + C_{i,2} = 1 | U_{i+1} = \mathbf{e}_5, C_{i,3} = 0) = 1,$$

and similarly as (21) that

$$P_{\omega}^0(C_{i,1} = 1 | U_{i+1} = \mathbf{e}_5, C_{i,3} = 0) = 1 - P_{\omega}^0(C_{i,2} = 1 | U_{i+1} = \mathbf{e}_5, C_{i,3} = 0) = \frac{\gamma_{i,1}}{\gamma_{i,1} + \gamma_{i,2}}.$$

Since the last step of excursion $\mathcal{B}_{i+1,2}$ is from $i-1$ to $i+1$, conditioned on $\{U_{i+1} = \mathbf{e}_5, C_{i,3} = 0\}$, after reaching i from below, it must jump down from i to $i-1$ or from i to $i-2$ at least one time, before the last step $\{i-1 \rightarrow i+1\}$ occurs. In the other words,

$$P_{\omega}^0(A_{i,3} + B_{i,3} = 1 | U_{i+1} = \mathbf{e}_5, C_{i,3} = 0) = 1$$

and

$$P_{\omega}^0(A_{i,3} = 1 | U_{i+1} = \mathbf{e}_5, C_{i,3} = 0) = 1 - P_{\omega}^0(B_{i,2} = 1 | U_{i+1} = \mathbf{e}_5, C_{i,3} = 0) = \frac{\alpha_{i,3}}{\alpha_{i,3} + \beta_{i,3}}.$$

On the other hand, conditioned on $\{U_{i+1} = \mathbf{e}_5, C_{i,3} = 0\}$, besides giving birthes to the above discussed particles, it may give birthes to a number of $\mathcal{A}_{i,1}$, $\mathcal{A}_{i,2}$, $\mathcal{B}_{i,1}$ and $\mathcal{B}_{i,2}$ particles.

Next we calculate the probabilities of $\{C_{i,3} = 1\}$ and $\{C_{i,3} = 0\}$ conditioned on $\{U_{i+1} = \mathbf{e}_5\}$.

One follows from the above discussions that the sum of the products of transition probabilities of all possible paths of an excursion $\mathcal{B}_{i+1,2}$ is $\beta_{i+1,2}$ and all those possible paths could be divided into two classes. Paths of the first class never visited i and paths of the second class visited i from below certain times. Sum of the product of transition probabilities of all possible first kind paths is $\gamma_{i,3}$ and that of the second kind paths is $\beta_{i+1,2} - \gamma_{i,3}$. Therefore,

$$P_{\omega}^0(C_{i,3} = 1 | U_{i+1} = \mathbf{e}_5) = 1 - P_{\omega}^0(C_{i,3} = 0 | U_{i+1} = \mathbf{e}_5) = \frac{\gamma_{i,3}}{\beta_{i+1,2}}.$$

Also the Markov property implies the independence of all excursions born to $\mathcal{B}_{i+1,2}$ particle at i . One follows by path decomposition and independence that

$$P_{\omega}^0(U_i = (0, \dots, 0, 1) | U_{i+1} = (0, 0, 0, 0, 1, 0, 0, 0, 0)) = \frac{\gamma_{i,3}}{\beta_{i+1,2}}, \quad (30)$$

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 1, c, d, 0, 1, 0, 0) | U_{i+1} = (0, 0, 0, 0, 1, 0, 0, 0, 0)) \\ = \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{(\beta_{i+1,2} - \gamma_{i,3}) \gamma_{i,1} \alpha_{i,3}}{\beta_{i+1,2} (\gamma_{i,1} + \gamma_{i,2}) (\alpha_{i,3} + \beta_{i,3})}, \end{aligned} \quad (31)$$

$$\begin{aligned} P_{\omega}^0(U_i = (a, b, 1, c, d, 0, 0, 1, 0) | U_{i+1} = (0, 0, 0, 0, 1, 0, 0, 0, 0)) \\ = \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{(\beta_{i+1,2} - \gamma_{i,3}) \gamma_{i,2} \alpha_{i,3}}{\beta_{i+1,2} (\gamma_{i,1} + \gamma_{i,2}) (\alpha_{i,3} + \beta_{i,3})}, \end{aligned} \quad (32)$$

$$\begin{aligned}
P_\omega^0(U_i = (a, b, 0, c, d, 1, 1, 0, 0) | U_{i+1} = (0, 0, 0, 0, 1, 0, 0, 0, 0)) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{(\beta_{i+1,2} - \gamma_{i,3}) \gamma_{i,1} \beta_{i,3}}{\beta_{i+1,2}(\gamma_{i,1} + \gamma_{i,2})(\alpha_{i,3} + \beta_{i,3})},
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
P_\omega^0(U_i = (a, b, 0, c, d, 1, 0, 1, 0) | U_{i+1} = (0, 0, 0, 0, 1, 0, 0, 0, 0)) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d (1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}) \frac{(\beta_{i+1,2} - \gamma_{i,3}) \gamma_{i,2} \beta_{i,3}}{\beta_{i+1,2}(\gamma_{i,1} + \gamma_{i,2})(\alpha_{i,3} + \beta_{i,3})}.
\end{aligned} \tag{34}$$

Summing up the discussions of this section, we have the following theorem, which has also been stated as Theorem 1.2 in the introduction section.

Theorem 3.1 $\{U_i\}_{i \leq 1}$ is a 9-type non-homogeneous branching process with immigration distribution as in (14), (15) and (16) above, and offsprings distributions as in (17-20), (22-25), (26-29) and (30-34) above.

Corollary 3.1 Let Q_i be a 9×9 matrix, whose l -th row are the means of number of particles born to a type- l particle of the $i+1$ -th generation. The matrices Q_i are called the mean matrices of the branching process $\{U_i\}_{i \leq 1}$. Let $x_i = \frac{\alpha_{i,1}}{1-\alpha_{i,1}-\alpha_{i,2}-\beta_{i,1}-\beta_{i,2}}$, $y_i = \frac{\alpha_{i,2}}{1-\alpha_{i,1}-\alpha_{i,2}-\beta_{i,1}-\beta_{i,2}}$, $z_i = \frac{\beta_{i,1}}{1-\alpha_{i,1}-\alpha_{i,2}-\beta_{i,1}-\beta_{i,2}}$, $w_i = \frac{\beta_{i,2}}{1-\alpha_{i,1}-\alpha_{i,2}-\beta_{i,1}-\beta_{i,2}}$, $1-v = \frac{\gamma_{i,3}}{\beta_{i+1,2}}$, $s_i = \frac{\alpha_{i,3}}{\alpha_{i,3}+\beta_{i,3}}$ and $t_i = \frac{\gamma_{i,1}}{\gamma_{i,1}+\gamma_{i,2}}$. Then one calculates from the branching mechanism of $\{U_i\}_{i \leq 1}$ that

$$Q_i = \begin{pmatrix} x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\ x_i & y_i & s_i & z_i & w_i & 1-s_i & 0 & 0 & 0 \\ x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\ x_i & y_i & 0 & z_i & w_i & 0 & t_i & 1-t_i & 0 \\ x_i v_i & y_i v_i & s_i v_i & z_i v_i & w_i v_i & (1-s_i)v_i & t_i v_i & (1-t_i)v_i & 1-v_i \\ x_i & y_i & 0 & z_i & w_i & 0 & t_i & 1-t_i & 0 \\ x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\ x_i & y_i & s_i & z_i & w_i & 1-s_i & 0 & 0 & 0 \\ x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

4 The ladder time T_1 and the branching process—Proof of Theorem 1.1

Recall that $T_1 = \inf\{n \geq 0 : X_n > 0\}$ is the first hitting time of $[1, \infty)$. In this section, we aim at expressing T_1 in terms of the multitype branching process $\{U_i\}_{i \leq 1}$.

Define

$$\begin{aligned}
D_{i,1} &= \#\{\text{steps by the walk from above } i-1 \text{ to } i-1 \text{ before time } T_1\}, \\
V_{i,1} &= \#\{\text{steps by the walk from } i-1 \text{ to } i \text{ before time } T_1\}, \\
V_{i,2} &= \#\{\text{steps by the walk from } i-2 \text{ to } i \text{ before time } T_1\}.
\end{aligned}$$

Then $\sum_{i \leq 0} D_{i,1}$ counts all steps jumping downward by the walk before time T_1 and $\sum_{i \leq 0} V_{i,1} + V_{i,2}$ counts all steps jumping upward by the walk before time T_1 . But both $\sum_{i \leq 0} D_{i,1}$ and $\sum_{i \leq 0} V_{i,1} + V_{i,2}$ did not count the last step by the walk hitting $[1, \infty)$. Therefore

$$T_1 = 1 + \sum_{i \leq 0} D_{i,1} + V_{i,1} + V_{i,2}.$$

This together with the fact $D_{i,1} = A_{i,1} + A_{i,2} + A_{i,3} + C_{i,1} + C_{i,2} + C_{i,3}$, $V_{i,1} = A_{i,1} + B_{i,1} + C_{i,1}$ and $V_{i,2} = A_{i,2} + B_{i,2} + C_{i,2}$ implies that

$$\begin{aligned} T_1 &= 1 + \sum_{i \leq 0} 2A_{i,1} + 2A_{i,2} + A_{i,3} + B_{i,1} + B_{i,2} + 2C_{i,1} + 2C_{i,2} + C_{i,3} \\ &= 1 + \sum_{i \leq 0} U_i(2, 2, 1, 1, 1, 0, 2, 2, 1)^T \end{aligned} \quad (36)$$

which proves Theorem 1.1.

Next we calculate the moment of T_1 by mean of the branching process.

One calculates from the immigration distributions (14-16) that

$$E_\omega^0(U_1) = \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}, \frac{\alpha_{1,3}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}, 0, \dots, 0 \right) =: u_1. \quad (37)$$

Therefore one follows from Markov property that, for $i \leq 0$,

$$E_\omega^0(U_i) = u_1 Q_0 \cdots Q_i.$$

Substituting to (36) one has that

$$E_\omega^0(T_1) = 1 + \sum_{i \leq 0} u_1 Q_0 \cdots Q_i(2, 2, 1, 1, 1, 0, 2, 2, 1)^T. \quad (38)$$

Remark 4.1 About the branching structure, we have the following remarks:

- The authors found that the branching structure could be simplified. Indeed, note that $\mathcal{C}_{i,j}$ has the same offspring distribution with $\mathcal{A}_{i,j}$, $j = 1, 2, 3$, and that $\mathcal{A}_{i,j}$ and $\mathcal{C}_{i,j}$, $j = 1, 2, 3$, play the same role in the (36). One could treat $\mathcal{A}_{i,j}$ and $\mathcal{C}_{i,j}$, $j = 1, 2, 3$, as the same type particles. In this point of view, a 6-type branching process is enough to count exactly all steps by the walk before T_1 . However we still use the original 9-type branching process because it is more understandable and each of the 9-type particles correspond to specific jump of the walk.
- While $R = 1$ the branching structure coincides with the one for (2-1) random walk constructed in Hong-Wang [3]. Indeed, for (2-1) model, there are only three type excursion, that is $\mathcal{A}_{i,1}$, $\mathcal{B}_{i,1}$ and $\mathcal{C}_{i,1}$. But as the above discussed, one could treat $\mathcal{A}_{i,1}$ and $\mathcal{C}_{i,1}$ particles as the same type. So one need only to consider a 2-type branching process $U_i = (A_{i,1}, B_{i,1})$. For (2-1) random walk, one follows easily that

$$\alpha_{i,1} = \omega_i(-1) \text{ and } \beta_{i,1} = \omega_i(-2).$$

Therefore, the distribution of $\mathcal{A}_{i+1,1}$ in (17) degenerates to

$$P_{\omega}^0(U_i = (a, b) | U_{i+1} = (1, 0)) = \frac{(a+b)!}{a!b!} (\omega_i(-1))^a (\omega_i(-2))^b \omega_i(1). \quad (39)$$

Since there is no jump of size 2 above, one see from Figure 5 that with probability 1, a type $\mathcal{C}_{i,1}$ particle will be born to a $\mathcal{B}_{i+1,1}$ particle. But we treat type $\mathcal{C}_{i,1}$ particle as type $\mathcal{A}_{i,1}$ particle now. Therefore the offspring distribution of $\mathcal{B}_{i,1}$ in (26) degenerates to

$$P_{\omega}^0(U_i = (a+1, b) | U_{i+1} = (0, 1)) = \frac{(a+b)!}{a!b!} (\omega_i(-1))^a (\omega_i(-2))^b \omega_i(1). \quad (40)$$

The offspring distributions in (39) and (40) coincides those offspring distributions in Hong-Wang [3].

- For (1-2) random walk, our result coincides with Hong-Zhang [4]. In this case, one needs only a 3-type branching process, that is $U_i = (A_{i,1}, A_{i,2}, A_{i,3})$. The offspring distributions of particle $\mathcal{A}_{i+1,1}$ and $\mathcal{A}_{i+1,3}$ in (17) and (18) degenerate to

$$P_{\omega}^0(U_i = (a, b, 0) | U_{i+1} = (1, 0, 0)) = \frac{(a+b)!}{a!b!} \alpha_{i,1}^a \alpha_{i,2}^b (1 - \alpha_{i,1} - \alpha_{i,2}), \quad (41)$$

$$P_{\omega}^0(U_i = (a, b, 0) | U_{i+1} = (0, 0, 1)) = \frac{(a+b)!}{a!b!} \alpha_{i,1}^a \alpha_{i,2}^b (1 - \alpha_{i,1} - \alpha_{i,2}). \quad (42)$$

For (1-2) random walk, one see from Figure 4 that, except generating offsprings as $\mathcal{A}_{i+1,1}$ particle, a type $\mathcal{A}_{i,2}$ particle gives birth with probability 1 to a type $\mathcal{A}_{i,3}$ particle. Therefore the offspring distribution in (22) degenerates to

$$P_{\omega}^0(U_i = (a, b, 1) | U_{i+1} = (0, 1, 0)) = \frac{(a+b)!}{a!b!} \alpha_{i,1}^a \alpha_{i,2}^b (1 - \alpha_{i,1} - \alpha_{i,2}). \quad (43)$$

One see that the above (41), (42) and (43) coincide with those offspring distributions in Hong-Zhang [4]. \square

5 An example for testing the branching structure

In this section, we let $\omega_0 = (q_2, q_1, p_1, p_2)$ where $p_1, p_2, q_1, q_2 > 0$ and $q_2 + q_1 + p_1 + p_2 = 1$ and let $\omega = (\dots, \omega_0, \omega_0, \omega_0, \dots)$. Now consider the random walk $\{X_n\}$ in the environment ω . Note that the transition probabilities of $\{X_n\}$ are now independent of the position.

We always assume that

$$p_1 + 2p_2 - q_1 - 2q_2 \geq 0 \quad (44)$$

which implies that $\limsup_{n \rightarrow \infty} X_n = \infty$. We also mention that for such degenerated ω , X_n could be realized by i.i.d. sum. Precisely, let $S_n := \sum_{i=1}^n \xi_i$, where $\{\xi_n\}$ is an independent sequence of random variables with common distribution $P(\xi_1 = 1) = p_1$, $P(\xi_1 = 2) = p_2$, $P(\xi_1 = -1) = q_1$ and $P(\xi_1 = -2) = q_2$. Then $\{X_n\} \stackrel{\mathcal{D}}{=} \{S_n\}$.

Let

$$M = \begin{pmatrix} -\frac{q_1+q_2}{q_2} & \frac{p_1+p_2}{q_2} & \frac{p_2}{q_2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One follows from (44) that M has three simple eigenvalues. Let f, g and h be the eigenvalues of M such that $|f| > |g| > |h|$. Then one follows also from (44) that $|f| > |g| \geq 1$ and $-1 < h < 0$.

Indeed, define $F(\lambda) := |\lambda - ME| = \lambda^3 + \frac{q_1+q_2}{q_2}\lambda^2 - \frac{p_1+p_2}{q_2}\lambda - \frac{p_2}{q_2}$. Then $F(0) = -\frac{p_2}{q_2} < 0$, and $F(-1) = \frac{1-q_2-p_2}{q_2} > 0$. Therefore $F(x)$ has a root $h \in (-1, 0)$. Note also that $F(y) > 0$ for y large enough and $F(1) = \frac{q_1+2q_2-(p_1+2p_2)}{q_2} \leq 0$. Then $F(\lambda)$ has a root in $[1, \infty)$. Also $F(-y) < 0$ for all y large enough. Since $F(-1) > 0$, $F(\lambda)$ has a root in $(-\infty, -1)$.

Note that the the exit probabilities in (5) degenerate to

$$\left\{ \begin{array}{l} \mathcal{P}_{b-1}(b) = \frac{\mathbf{e}_1 M^{b-a-1} [\mathbf{e}_2 - \mathbf{e}_3]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} l \mathbf{e}_1^T \right) - \mathbf{e}_1 M^{b-a-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} [\mathbf{e}_2 - \mathbf{e}_3]^T}{\mathbf{e}_1 M^{b-a-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 M^{b-a-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} \mathbf{e}_1^T \right)}, \\ \mathcal{P}_{b-2}(b) = \frac{\mathbf{e}_1 M^{b-a-1} [\mathbf{e}_2 - \mathbf{e}_3]^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 M^{b-a-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} [\mathbf{e}_2 - \mathbf{e}_3]^T}{\mathbf{e}_1 M^{b-a-1} \mathbf{e}_1^T \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} [\mathbf{e}_1 - \mathbf{e}_2]^T - \mathbf{e}_1 M^{b-a-1} [\mathbf{e}_1 - \mathbf{e}_2]^T \left(1 + \sum_{l=a+1}^{b-1} \mathbf{e}_1 M^{b-l} \mathbf{e}_1^T \right)}. \end{array} \right. \quad (45)$$

Letting $a \rightarrow -\infty$, by some careful calculations, one follows from (45) that

$$\begin{aligned} f_{b-1}(1) &:= \lim_{a \rightarrow -\infty} \mathcal{P}_{b-1}(b) = 1 + h; \\ f_{b-2}(1) &:= \lim_{a \rightarrow -\infty} \mathcal{P}_{b-2}(b) = 1 + h + h^2. \end{aligned} \quad (46)$$

We remark that, for $g = 1$, and $g \neq 1$, it is a bit different to find the limit of (45) as $a \rightarrow -\infty$. But the limits have the same form for both $g = 1$ and $g \neq 1$.

Recall that $\mathcal{P}_{b-1}(b)$ is the simplification of $\mathcal{P}_{b-1}(a, b, b)$ which by definition equals to

$$P_\omega^{b-1}(\text{the walk exits the interval } [a+1, b-1] \text{ at } b).$$

Therefore one sees from (46) that

$$\begin{aligned} f_{b-1}(1) &= P_\omega^{b-1}(\text{the walk exits the interval } (-\infty, b-1] \text{ at } b) = 1 + h; \\ f_{b-2}(1) &= P_\omega^{b-2}(\text{the walk exits the interval } (-\infty, b-1] \text{ at } b) = 1 + h + h^2. \end{aligned}$$

Similarly as all M_k degenerate to M , one follows by some careful calculations from (6) that

$$\begin{aligned} f_{b-1}(2) &= P_\omega^{b-1}(\text{the walk exits the interval } (-\infty, b-1] \text{ at } b+1) = -h; \\ f_{b-2}(2) &= P_\omega^{b-2}(\text{the walk exits the interval } (-\infty, b-1] \text{ at } b+1) = -h - h^2. \end{aligned}$$

Recall that $T_1 := \inf\{n > 0 : X_n > 0\}$. Then

$$f_0(1) = P_\omega^0(X_{T_1} = 1) = 1 + h$$

and

$$f_0(2) = P_\omega^0(X_{T_1} = 2) = -h.$$

Therefore one has that

$$E_\omega^0(X_{T_1}) = 1f_0(1) + 2f_0(2) = 1 - h. \quad (47)$$

Since $h \in (-1, 0)$, $1 < E_\omega^0(X_{T_1}) < 2$, which is also natural by intuition.

One should note that the above calculations of the mean of X_{T_1} involve only the exit probabilities of the walk from $(-\infty, 0]$.

On the other hand, recall that T_1 could be expressed by the 9-type branching process constructed in Section 3. That is,

$$T_1 = 1 + \sum_{i \leq 0} U_i(2, 2, 1, 1, 1, 0, 2, 2, 1)^T.$$

Since $\omega_i = \omega_0$ for all i , we omit the subscript “ i ” in the notation related. For example we write $\alpha_{i,1}$ as α_1 , w_i as w et al. One has the following results: $\alpha_1 = -q_1 p_1 h / p_2$, $\alpha_2 = q_1(p_2 + h p_1 + h p_2) / p_2$, $\alpha_3 = -q_1 h$; $\beta_1 = -q_2 p_1 h(1+h) / p_2$, $\beta_2 = q_2(p_2 + (p_1 + p_2)h(1+h)) / p_2$, $\beta_3 = -q_2 h(1+h)$; $\gamma_1 = -q_2 p_1 h / p_2$, $\gamma_2 = q_2(p_2 + h p_1 + h p_2) / p_2$, $\gamma_3 = -q_2 h$; $x = q_1 p_1 h^2 / p_2^2$, $y = -q_1 h(p_2 + h p_1 + h p_2) / p_2^2$, $z = q_2 p_1 h^2(1+h) / p_2^2$, $w = -q_2 h^3(1 + q_2 h) / p_2^2$; $v = 1 + p_2 / (h(1 + q_2 h))$, $s = q_1 / (q_1 + q_2(1 + h))$, $t = -p_1 h / (p_2(1 + h))$; $u_1 = (-p_1 h / p_2, (p_2 + p_1 h + p_2 h) / p_2, -h p_2 / p_2, 0, 0, 0, 0, 0, 0)$. With the above notations, the mean matrix Q_i of the branching process $\{U_i\}$ in (35) degenerates to

$$Q = \begin{pmatrix} x & y & 0 & z & w & 0 & 0 & 0 & 0 \\ x & y & s & z & w & 1-s & 0 & 0 & 0 \\ x & y & 0 & z & w & 0 & 0 & 0 & 0 \\ x & y & 0 & z & w & 0 & t & 1-t & 0 \\ xv & yv & sv & zv & wv & (1-s)v & tv & (1-t)v & 1-v \\ x & y & 0 & z & w & 0 & t & 1-t & 0 \\ x & y & 0 & z & w & 0 & 0 & 0 & 0 \\ x & y & s & z & w & 1-s & 0 & 0 & 0 \\ x & y & 0 & z & w & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (48)$$

Then one has from (38) that

$$E_\omega^0(T_1) = 1 + \sum_{i \leq 0} u_1 Q^{i+1}(2, 2, 1, 1, 1, 0, 2, 2, 1)^T.$$

Note that if one assumes

$$E_\omega^0(X_1) = p_1 + 2p_2 - q_1 - 2q_2 > 0,$$

then Ward Equation

$$E_\omega^0(X_{T_1}) = E_\omega^0(T_1)E_\omega^0(X_1) \quad (49)$$

should hold.

Recall that $E_\omega^0(X_{T_1})$ was calculate in (47) by the exit probability and that $E_\omega^0(T_1)$ was calculated by the branching structure constructed in Section 3. Therefore it will provide a good testification of the branching structure to show that Ward Equation (49) holds. Equivalently, to show (49), one needs only to show

$$1 + \sum_{i \leq 0} u_1 Q^{i+1}(2, 2, 1, 1, 1, 0, 2, 2, 1)^T = \frac{1 - h}{p_1 + 2p_2 - q_1 - 2q_2}. \quad (50)$$

Some elementary calculation shows that Q has four nonzero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and all other eigenvalues are 0. One could find the eigenvectors to get a matrix B such that

$$Q = B\Lambda B^{-1} \quad (51)$$

with

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_4 & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \end{pmatrix}.$$

Theoretically, one could substitute (51) to the left-hand side of (50) to show that (50) does hold. But the calculations are technical and tedious. Instead of giving such tedious calculations, some numerical test may be preferred. By the “Matlab” we get the following table.

| Test of the branching structure | | | | | | | |
|---------------------------------|--------|--------|--------|-------------------|--|--|--------------|
| p_1 | p_2 | q_1 | q_2 | $E_\omega^0(X_1)$ | $E_\omega^0(X_{T_1})$:= $E_\omega^0(T_1)E_\omega^0(X_1)$ | $E_\omega^0(X_{T_1})$:= $f_0(1) + 2f_0(2)$ | Error |
| 0.2100 | 0.3500 | 0.3600 | 0.0800 | 0.3900 | 1.467727692 | 1.467727692 | -6.6613e-016 |
| 0.3000 | 0.2100 | 0.3000 | 0.1900 | 0.0400 | 1.323718710 | 1.323718710 | 2.8644e-014 |
| 0.1789 | 0.3211 | 0.1801 | 0.3199 | 0.0012 | 1.481684406 | 1.481684406 | 1.8585e-013 |
| 0.4998 | 0.0002 | 0.4999 | 0.0001 | 0.0001 | 1.000399840 | 1.000399840 | -3.5456e-011 |
| 0.3627 | 0.1373 | 0.3628 | 0.1372 | 0.0001 | 1.226498171 | 1.226490265 | 7.9058e-006 |

The column “ $E_\omega^0(X_{T_1}) = E_\omega^0(T_1)E_\omega^0(X_1)$ ” of the table means that

$$E_\omega^0(X_{T_1}) = E_\omega^0(T_1)E_\omega^0(X_1) = E_\omega^0(T_1)(p_1 + 2p_2 - q_1 - 2q_2),$$

where $E_\omega^0(T_1)$ is calculated by the multitype branching process $\{U_i\}$, that is,

$$E_\omega^0(T_1) = 1 + \sum_{i \leq 0} u_i Q^{i+1}(2, 2, 1, 1, 1, 0, 2, 2, 1)^T.$$

6 Invariant measure equation and law of large numbers of (2-2) RWRE—Proof of Theorem 1.3

In this section, we consider random walk $\{X_n\}$ in random environment ω . Since by assumption $E^0(T_1) < \infty$, then P^0 -a.s., $T_1 < \infty$.

Define $\bar{\omega}(n) = \theta^{X_n}\omega$. The process $\{\bar{\omega}(n)\}$ is called the environment viewed from particles. One easily show that $\{\bar{\omega}(n)\}$ is indeed a Markov process under either P_ω^0 or P^0 , with transitional kernel

$$K(\omega, d\omega') = \omega_0(2)\delta_{\theta^2\omega=\omega'} + \omega_0(1)\delta_{\theta\omega=\omega'} + \omega_0(-1)\delta_{\theta^{-1}\omega=\omega'} + \omega_0(-2)\delta_{\theta^{-2}\omega=\omega'}.$$

It is important to find the invariant measure and the corresponding invariant density for the transition kernel $K(\omega, d\omega')$. If one has the invariant density in the hand, then one could show the law of large numbers for random walk in random environment $\{X_n\}$ and the limit velocity of the transient walk could be expressed by the invariant density.

We borrow some notations from [4] to give the invariant measure. Define $\varphi_{\theta^k\omega}^1 = P_{\theta^k\omega}^0(X_{T_1} = 1)$ and $\varphi_{\theta^k\omega}^2 = P_{\theta^k\omega}^0(X_{T_1} = 2)$. Whenever $E^0(T_1) < \infty$, define

$$Q(d\omega) = E^0 \left(\frac{1_{X_{T_1}=1}}{\varphi_{\omega}^1} \sum_{i=0}^{T_1-1} 1_{\overline{\omega}(i) \in d\omega} + \frac{1_{X_{T_1}=2}}{\varphi_{\omega}^2} \sum_{i=0}^{T_1-1} 1_{\overline{\omega}(i) \in d\omega} \right) \text{ and } \overline{Q}(d\omega) = Q(d\omega)/Q(\Omega).$$

Then one follows verbatim as [4] that the measure Q is invariant under transition kernel $K(\omega, d\omega')$. Precisely, one has that, for $B \in \mathcal{F}$,

$$Q(B) = \iint 1_{\omega' \in B} K(\omega, d\omega') Q(d\omega).$$

Also, following [4] one shows that

$$\frac{dQ}{dP} = \sum_{i \leq 0} E_{\theta^{-i}\omega}^0(N_i | X_{T_1} = 1) + E_{\theta^{-i}\omega}^0(N_i | X_{T_1} = 2), \quad (52)$$

where $N_i := \#\{k \in [0, T_1) : X_k = i\}$.

The branching structure enables us to calculate the right-hand side of (52) and give specifically $\frac{dQ}{dP}$. In fact, note that for $i \leq -2$,

$$\begin{aligned} N_i = & A_{i+1,1} + A_{i+1,2} + A_{i+1,3} + C_{i+1,1} + C_{i+1,2} + C_{i+1,3} \\ & + A_{i,1} + A_{i,2} + B_{i,1} + B_{i,2} + C_{i,1} + C_{i,2}. \end{aligned}$$

Then

$$\begin{aligned} E_{\omega}^0(N_i | X_{T_1} = 1) = & \frac{E_{\omega}^0(U_{i+1}(1, 1, 1, 0, 0, 0, 1, 1, 1)^T; X_{T_1} = 1)}{P_{\omega}^0(X_{T_1} = 1)} \\ & + \frac{E_{\omega}^0(U_i(1, 1, 0, 1, 1, 0, 1, 1, 0)^T; X_{T_1} = 1)}{P_{\omega}^0(X_{T_1} = 1)}. \end{aligned} \quad (53)$$

Temporally, we set $\mathbf{v}_1 = (1, 1, 1, 0, 0, 0, 1, 1, 1)^T$ and $\mathbf{v}_2 = (1, 1, 0, 1, 1, 0, 1, 1, 0)^T$. The first term in the right-hand side of (53) equals to

$$\begin{aligned} & \frac{1}{f_0(1)} E_{\omega}^0(U_{i+1}\mathbf{v}_1; A_{1,1} = 1) + \frac{1}{f_0(1)} E_{\omega}^0(U_{i+1}\mathbf{v}_1; A_{1,2} = 1) \\ & = \frac{\omega_1(-1)}{\alpha_{1,1} + \alpha_{1,2}} \left(E_{\omega}^0(U_{i+1}\mathbf{v}_1 | A_{1,1} = 1) P_{\omega}^0(A_{1,1} = 1) + E_{\omega}^0(U_{i+1}\mathbf{v}_1 | A_{1,2} = 1) P_{\omega}^0(A_{1,2} = 1) \right) \\ & = \frac{\omega_1(-1)}{\alpha_{1,1} + \alpha_{1,2}} \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}} \mathbf{e}_1 Q_0 \cdots Q_{i+1}\mathbf{v}_1 + \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}} \mathbf{e}_2 Q_0 \cdots Q_{i+1}\mathbf{v}_1 \right) \\ & = \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) Q_0 \cdots Q_{i+1}\mathbf{v}_1, \end{aligned}$$

since $f_0(1) = (\alpha_{1,1} + \alpha_{1,2})/\omega_1(-1)$ and $\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3} = \omega_1(-1)$.

Similarly the second term in the right-hand side of (53) equals to

$$\left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) Q_0 \cdots Q_i\mathbf{v}_2.$$

Therefore,

$$E_{\omega}^0(N_i | X_{T_1} = 1) = \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) (Q_0 \cdots Q_{i+1}\mathbf{v}_1 + Q_0 \cdots Q_i\mathbf{v}_2). \quad (54)$$

One the other hand,

$$E_\omega^0(N_i|X_{T_1} = 2) = E_\omega^0(N_i|A_{1,1} = 2) = \mathbf{e}_3(Q_0 \cdots Q_{i+1}\mathbf{v}_1 + Q_0 \cdots Q_i\mathbf{v}_2).$$

Next since $N_0 = 1 + A_{0,1} + A_{0,2} + B_{0,1} + B_{0,2}$, one has that

$$\begin{aligned} E_\omega^0(N_0|X_{T_1} = 2) &= E_\omega^0(N_0|A_{1,3} = 1) \\ &= 1 + \mathbf{e}_3 Q_0(1, 1, 0, 1, 1, 0, 0, 0)^T = 1 + \mathbf{e}_3 Q_0 \mathbf{v}_2; \\ E_\omega^0(N_0|X_{T_1} = 1) &= 1 + \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) Q_0(1, 1, 0, 1, 1, 0, 0, 0)^T \\ &= 1 + \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) Q_0 \mathbf{v}_2, \end{aligned} \quad (55)$$

where the second equality holds due to the special structure of the mean matrix Q_0 . Note that

$$N_{-1} = A_{0,1} + A_{0,2} + A_{0,3} + A_{-1,1} + A_{-1,2} + B_{-1,1} + B_{-1,2} + C_{-1,1} + C_{-1,2},$$

where we mention that if the last step before T_1 is $\{-1 \rightarrow 1\}$, $A_{0,3} = 1$, otherwise, $A_{0,3} = 0$.

Similarly as above, one has that

$$\begin{aligned} E_\omega^0(N_{-1}|X_{T_1} = 1) &= \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) (Q_0(1, 1, 1, 0, \dots, 0)^T + Q_0 Q_{-1} \mathbf{v}_2) \\ &= \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 0, \dots, 0 \right) (Q_0(1, 1, 1, 0, 0, 0, 1, 1)^T + Q_0 Q_{-1} \mathbf{v}_2); \\ E_\omega^0(N_{-1}|X_{T_1} = 2) &= E_\omega^0(N_{-1}|A_{1,3} = 1) = \mathbf{e}_3(Q_0(1, 1, 1, 0, \dots, 0)^T + Q_0 Q_{-1} \mathbf{v}_2) \\ &= E_\omega^0(N_{-1}|A_{1,3} = 1) = \mathbf{e}_3(Q_0(1, 1, 1, 0, 0, 0, 1, 1)^T + Q_0 Q_{-1} \mathbf{v}_2) \end{aligned} \quad (56)$$

Substituting (54), (55) and (56) to (52) one concludes that

$$\begin{aligned} \frac{dQ}{dP} &= 2 + \sum_{i=0}^{\infty} \left(\frac{\alpha_{1+i,1}}{\alpha_{1+i,1} + \alpha_{1+i,2}}, \frac{\alpha_{1+i,1}}{\alpha_{1+i,1} + \alpha_{1+i,2}}, 1, 0, \dots, 0 \right) Q_i \cdots Q_0(1, 1, 0, 1, 1, 0, 1, 1, 0)^T \\ &\quad + \sum_{i=1}^{\infty} \left(\frac{\alpha_{1+i,1}}{\alpha_{1+i,1} + \alpha_{1+i,2}}, \frac{\alpha_{1+i,1}}{\alpha_{1+i,1} + \alpha_{1+i,2}}, 1, 0, \dots, 0 \right) Q_i \cdots Q_1(1, 1, 1, 0, 0, 0, 1, 1, 1)^T \\ &=: \Pi(\omega). \end{aligned} \quad (57)$$

We mention that the purpose of deriving the invariant measure $Q(d\omega)$ and the invariant density dQ/dP is to prove a law of large number for the (2-2) random walk in random environment by an approach known as “the environment viewed from particles”.

One follows similarly as Zeitouni [7] Corollary 2.1.25 that $\{\bar{\omega}(n)\}$ is stationary and ergodic under the measure $\bar{Q} \otimes P_\omega^0$. Define the local drift at site x in the environment ω as $d(x, \omega) = E_\omega^x(X_1 - x)$. The ergodicity of $\{\omega(n)\}$ under $Q \otimes P_\omega^0$ implies that:

$$\frac{1}{n} \sum_{k=0}^{n-1} d(X_k, \omega) = \frac{1}{n} \sum_{k=0}^{n-1} d(0, \bar{\omega}(k)) \xrightarrow{n \rightarrow \infty} E_{\bar{Q}}(d(0, \omega)) \quad \bar{Q} \otimes P_\omega^0\text{-a.s.}$$

But

$$\begin{aligned} X_n &= \sum_{i=1}^n (X_i - X_{i-1}) = \sum_{i=1}^n (X_i - X_{i-1} - d(X_i, \omega)) + \sum_{i=1}^n d(X_i, \omega) \\ &=: M_n + \sum_{i=1}^n d(X_i, \omega). \end{aligned}$$

Similarly as Zeitouni [7], $\{M_n\}$ is a martingale and P^0 -a.s.,

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} X_n = E_{\overline{Q}}(d(0, \omega)) =: V_P.$$

Next we calculate V_P . Note that

$$\begin{aligned} V_P &= E_{\overline{Q}}(d(0, \omega)) = E_{\overline{Q}}(X_1) = E_{\mathbb{P}}\left(\Pi(\omega)(2\omega_0(-2) + \omega_0(-1) + \omega_0(1) + 2\omega_0(2))\right) / Q(\Omega) \\ &= \frac{E_{\mathbb{P}}\left(\Pi(\omega)(2\omega_0(-2) + \omega_0(-1) + \omega_0(1) + 2\omega_0(2))\right)}{E^0(T_1 | X_{T_1} = 1) + E^0(T_1 | X_{T_1} = 2)} \end{aligned} \quad (58)$$

But the denominator in (58) equals to

$$E_{\mathbb{P}}\left(2 + \left(\frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 1, 0, \dots, 0\right) \sum_{i \leq 0} Q_0 \cdots Q_i(2, 2, 1, 1, 1, 0, 2, 2, 1)^T\right) := E_{\mathbb{P}}(D(\omega)). \quad (59)$$

Substituting to (58) one has that

$$V_P = \frac{E_{\mathbb{P}}\left(\Pi(\omega)(2\omega_0(-2) + \omega_0(-1) + \omega_0(1) + 2\omega_0(2))\right)}{E_{\mathbb{P}}(D(\omega))}.$$

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